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# Quantum Fields on Star Graphs

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**Abstract**

We construct canonical quantum fields which propagate on a star graph modeling a quantum wire. The construction uses a deformation of the algebra of canonical commutation relations, encoding the interaction in the vertex of the graph. We discuss in this framework the Casimir effect and derive the correction to the Stefan-Boltzmann law induced by the vertex interaction. We also generalize the algebraic setting in order to cover systems with integrable bulk interactions and solve the quantum non-linear Schrödinger model on a star graph.

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# 1 Introduction

The physics of thin three-dimensional graph-like structures [1], two of whose dimensions being of the order of a few nanometers, attracts recently much attention. Due to the impressive progress in nanotechnology such devices, called quantum wires, can be constructed and tested nowadays in laboratory. At the theoretical side, the quantum behavior of these essentially one-dimensional systems poses interesting mathematical and physical problems, which have been addressed in the past decade by various authors [2]-[9]. In the present paper we pursue further the study of quantum wires, developing field theory on such structures. For this purpose we adapt the algebraic approach developed in [10]-[12] for dealing with systems with boundaries and point-like defects. This approach translates the boundary value problem at hand in algebraic terms and offers an efficient framework for the construction of quantum fields.

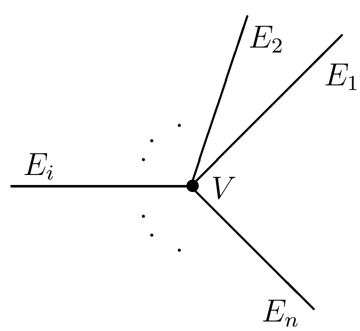


Figure 1: A star graph $G$ with $n$ edges.

We focus in this paper on quantum wires of the form of a star graph $G$ shown in Fig. 1, where $V$ stands for the vertex of the graph and $E_i$ (with $i = 1, ..., n$) denote its edges. Being the fundamental building blocks for general graphs, the star graphs play a distinguished role [3, 4] in the subject. The first question, we address in section 2, is the explicit construction of a canonical scalar quantum field, freely propagating in the bulk $B \equiv G \setminus V$ of the graph $G$. The vertex is considered in our approach as a defect (boundary in the case $n = 1$) interacting with the field and characterized by reflection ($n \geq 1$) and transmission ($n \geq 2$) coefficients. We consider the case without dissipation and treat both relativistic and non-relativistic dispersion relations. In section 3 we derive the correlation functions with respect to two different cyclic states - the Fock vacuum and a Gibbs state at finite (inverse) temperature $\beta$. Some basic physical observables are analyzed in section 4. In particular, we compute there the Casimir energy and the correction to the Stefan-Boltzmann law generated by the interaction localized in the vertex of the graph. Inspired by factorized scattering in 1+1 dimensional integrable systems, we introduce in section 5 a two-body interaction in the bulk $B$. The system treated here is the non-linear Schrödinger model (non-relativistic $\psi^4$ theory) on a star graph. We establish the exact operator solution of this model. Section 6 contains our conclusions and some ideas for future developments.

# 2 Quantum fields on a star graph

Consider a star graph $G$. Each point $P$ in the bulk $B$ of $G$ can be parametrized by a pair $(i,x)$ where $i=1,...,n$ indicates the edge and $x>0$ the distance of $P$ from the vertex $V$ along the edge. We will see below that this choice of coordinates ensures a uniform treatment of all edges as far as the reflection and transmission by the vertex $V$ is concerned. The embedding of $G$ and the relative position of its edges $E_i$ in the ambient space are irrelevant in what follows.

## 2.1 Non-relativistic dispersion

We start with the non-relativistic complex scalar field $\psi(t,x,i)$ satisfying

$$\left(\mathrm{i}\partial_t + \frac{1}{2m}\partial_x^2\right)\psi_i(t,x) = 0\,, \qquad t\in\mathbb{R}\,,\quad x>0\,,\quad i=1,...,n\,, \tag{2.1}$$

with standard initial conditions given by the equal-time canonical commutation relations

$$[\psi_{i_1}(0,x_1)\,,\,\psi_{i_2}(0,x_2)] = [\psi^{*i_1}(0,x_1)\,,\,\psi^{*i_2}(0,x_2)] = 0\,, \tag{2.2}$$

$$[\psi_{i_1}(0,x_1)\,,\,\psi^{*i_2}(0,x_2)] = \delta_{i_1}^{i_2}\,\delta(x_1-x_2)\,, \tag{2.3}$$

where $*$ denotes Hermitian conjugation. Since eqs. (2.1-2.3) hold only in the bulk $B$, they do not fix uniquely the solution. For this purpose one needs some boundary conditions in the vertex $V$. Setting

$$\psi_i(t,0) = \lim_{\substack{x\to 0\\ x>0}}\psi_i(t,x)\,, \qquad (\partial_x\psi_i)(t,0) = \lim_{\substack{x\to 0\\ x>0}}(\partial_x\psi_i)(t,x)\,, \tag{2.4}$$

we require [3] that[3]

$$A_i^j\psi_j(t,0) + B_i^j(\partial_x\psi_j)(t,0) = 0\,, \quad \forall\, t\in\mathbb{R}\,,\quad i=1,...,n\,, \tag{2.5}$$

[3]Summation over repeated upper and lower indices is understood, unless otherwise stated.

$A$ and $B$ being two complex $n \times n$ matrices. It has been shown in [3], that (2.5) uniquely defines a self-adjoint extension of the operator $-\partial_x^2$ on $G$, provided that the composite matrix $(A, B)$ has a rank $n$ and

$$A\, B^* - B\, A^* = 0\,. \tag{2.6}$$

Moreover, this extension admits a non-trivial scattering matrix, reading in momentum space

$$S(k) = -(A + \mathrm{i}kB)^{-1}(A - \mathrm{i}kB)\,, \tag{2.7}$$

which, in view of (2.6), can be written also in the form

$$S(k) = -(A^* - \mathrm{i}kB^*)(AA^* + k^2BB^*)^{-1}(A - \mathrm{i}kB)\,. \tag{2.8}$$

The entries of $S$ have a simple physical interpretation: the diagonal element $S_i^i(k)$ is the reflection amplitude on the edge $E_i$, whereas $S_i^j(k)$ with $i \neq j$ is the transmission amplitude from $E_i$ to $E_j$. From (2.7) and (2.8) one can easily deduce that $S$ is unitary

$$S(k)^* = S(k)^{-1}\,, \tag{2.9}$$

and satisfies Hermitian analyticity

$$S(k)^* = S(-k)\,. \tag{2.10}$$

Combining (2.9) and (2.10) one gets

$$S(k)\, S(-k) = \mathbb{I}_n\,, \tag{2.11}$$

where $\mathbb{I}_n$ is the $n \times n$ identity matrix. The property (2.11) is essential in what follows.

Turning back to the quantum field $\psi$, we introduce the wave functions [3]

$$\chi_i^j(x;k) \equiv \mathrm{e}^{\mathrm{i}kx}\delta_i^j + S_i^j(-k)\mathrm{e}^{-\mathrm{i}kx}\,, \qquad k > 0\,, \tag{2.12}$$

which are orthogonal (see the appendix)

$$\int_0^{+\infty} dx\, \chi_i^{*l}(x;k)\chi_l^j(x;p) = \delta_i^j\, 2\pi\, \delta(k-p)\,. \tag{2.13}$$

If in addition

$$\check{S}_i^j(x) \equiv \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\mathrm{e}^{\mathrm{i}kx} S_i^j(k) = 0\,, \qquad x > 0\,, \tag{2.14}$$

then $\chi$ form a complete set, i.e.

$$\int_0^{+\infty} \frac{dk}{2\pi}\, \chi^{*l}_i(x;k)\chi^j_l(y;k) = \delta^j_i\, \delta(x-y)\,. \tag{2.15}$$

At this point we define

$$\psi_i(t,x) = \int_0^{+\infty} \frac{dk}{2\pi} \mathrm{e}^{-\mathrm{i}\omega(k)t} \chi^j_i(x;k) a_j(k)\,, \tag{2.16}$$

$$\psi^{*i}(t,x) = \int_0^{+\infty} \frac{dk}{2\pi} \mathrm{e}^{\mathrm{i}\omega(k)t} a^{*j}(k) \chi^{*i}_j(x;k)\,, \tag{2.17}$$

where $\omega(k)$ is the dispersion relation

$$\omega(k) = \frac{k^2}{2m}\,, \qquad m > 0\,, \tag{2.18}$$

and $\{a_i(k),\, a^{*i}(k)\, :\, k > 0\}$ generate an associative algebra $\mathcal{A}_+$ with identity element $\mathbf{1}$ and satisfy the canonical commutation relations

$$a_{i_1}(k_1)\, a_{i_2}(k_2) - a_{i_2}(k_2)\, a_{i_1}(k_1) = 0\,, \tag{2.19}$$

$$a^{*i_1}(k_1)\, a^{*i_2}(k_2) - a^{*i_2}(k_2)\, a^{*i_1}(k_1) = 0\,, \tag{2.20}$$

$$a_{i_1}(k_1)\, a^{*i_2}(k_2) - a^{*i_2}(k_2)\, a_{i_1}(k_1) = 2\pi\delta(k_1-k_2)\, \delta^{i_2}_{i_1}\, \mathbf{1}\,. \tag{2.21}$$

The equation of motion (2.1) and the boundary condition (2.5) follow directly from the representation (2.16). The commutation relations (2.2,2.3) are a consequence of (2.19-2.21) and the condition (2.14). If the latter is violated, in addition to $\{a_i(k),\, a^{*i}(k)\, :\, k > 0\}$ the system admits other degrees of freedom, which must be taken into account for reproducing (2.3). These new degrees correspond to vertex bound states, which have been described in detail for $n = 2$ in [13]. In order to simplify the discussion, we assume in the present paper that (2.14) holds.

In the representation (2.16,2.17) the boundary condition (2.5) is captured by the wave functions $\chi$ and $\chi^*$. Remarkably enough, this information can be shifted to $a_i(k)$ and $a^{*i}(k)$, extending $\mathcal{A}_+$ to any $k \in \mathbb{R}$ and considering the algebra $\mathcal{A}$ generated by $\{a_i(k),\, a^{*i}(k)\, :\, k \in \mathbb{R}\}$ which satisfy (2.19,2.20),

$$a_{i_1}(k_1)\, a^{*i_2}(k_2) - a^{*i_2}(k_2)\, a_{i_1}(k_1) = 2\pi\delta(k_1-k_2)\, \delta^{i_2}_{i_1}\, \mathbf{1} + S^{i_2}_{i_1}(k_1) 2\pi\delta(k_1+k_2)\, \mathbf{1}\,, \tag{2.22}$$

and the constraints

$$a_i(k) = S^j_i(k) a_j(-k)\,, \qquad a^{*i}(k) = a^{*j}(-k) S^i_j(-k)\,. \tag{2.23}$$

One can easily verify in fact that in terms of $\mathcal{A}$ the fields $\psi$ and $\psi^*$ take the simpler form

$$\psi_i(t,x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} a_i(k) \mathrm{e}^{-\mathrm{i}\omega(k)t+\mathrm{i}kx}\, , \tag{2.24}$$

$$\psi^{*i}(t,x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} a^{*i}(k) \mathrm{e}^{\mathrm{i}\omega(k)t-\mathrm{i}kx}\, , \tag{2.25}$$

where now the integration over $k$ runs over the whole line. The constraints (2.23) implement in (2.24,2.25) the boundary conditions (2.5) and are consistent because of (2.11). Hermitian analyticity (2.10) implies that the mapping

$$I \ :\ a^{*i}(k) \mapsto a_i(k)\, , \qquad I \ :\ a_i(k) \mapsto a^{*i}(k)\, , \tag{2.26}$$

generates an involution in $\mathcal{A}$, which is consistent with the second term in the right hand side of (2.22). Because of (2.14), the initial conditions (2.2,2.3) are not influenced by this term. It contributes however to the time evolution and codifies the interaction in the vertex $V$, playing the role of a defect. Indeed, the algebra $\mathcal{A}$ defined by (2.19,2.20,2.22, 2.23) is a special case of the boundary and the reflection-transmission algebras introduced in [10] and [11, 12] for dealing respectively with boundaries and defects in quantum field theory. With our present choice of coordinates, the direction of both reflected and transmitted waves coincides with the orientation of the edges. For this reason both reflection and transmission amplitudes in (2.22) multiply the $\delta(k_1+k_2)$-contribution.[4]

The Hamiltonian $H$ of our system is the self-adjoint extension of $-\partial_x^2$, defined by the pair $(A,B)$. The work with this somehow implicit form of $H$ is greatly simplified in the algebraic setting, where $H$ takes the familiar quadratic form

$$H = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \omega(k) a^{*i}(k) a_i(k) = \int_0^{+\infty} \frac{dk}{2\pi} \omega(k) a^{*i}(k) a_i(k)\, , \tag{2.27}$$

which is very convenient to deal with. One easily verifies for instance

$$\psi_i(t,x) = \mathrm{e}^{\mathrm{i}tH} \psi_i(0,x) \mathrm{e}^{-\mathrm{i}tH}\, , \tag{2.28}$$

which confirms that (2.27) generates indeed the time evolution of $\psi$. It is also worth stressing that the energy is conserved, showing that the vertex $V$ behaves like a defect without dissipation.

We are interested in the fields $\psi$ and $\psi^*$ in the range $x \geq 0$. Nevertheless, eqs.(2.24,2.25) keep a well-defined meaning for any $x \in \mathbb{R}$. The resulting fields

[4] This is not the case for the coordinates adopted for $n=2$ in [12], where the transmission coefficients multiply $\delta(k_1-k_2)$ because $E_1 = \mathbb{R}_+$ but $E_2 = \mathbb{R}_-$.

satisfy the equation of motion (2.1), but do not obey the equal-time commutator (2.3) on the whole line. The constraints (2.23) imply

$$\psi_i(t,x) = \int_{-\infty}^{0} dy\, \check{S}_i^j(y)\, \psi_j(t, y-x)\,, \tag{2.29}$$

which relates the values of $\psi$ on $\mathbb{R}_+$ with those on $\mathbb{R}_-$.

Summarizing, we translated so far the construction [3] of the self-adjoint extension of the operator $-\partial_x^2$, relative to the boundary condition (2.5), in terms of the algebra $\mathcal{A}$. The advantage of the algebraic approach is that:

(i) it admits a straightforward extension to more general scattering matrices then those defined by (2.7);

(ii) it provides a framework for introducing and studying some integrable interactions in the bulk of $G$.

Let us concentrate now on point (i), dedicating to (ii) section 5 below. The key observation is that the algebra $\mathcal{A}$ is actually well-defined for any $S$-matrix satisfying (2.9,2.10), which allows to consider much more general boundary conditions then (2.5). This fact is evident already for $n = 1$. In this case the general $S$-matrix satisfying (2.9,2.10) can be written in the form

$$S(k) = \frac{s(k) - \mathrm{i}}{s(k) + \mathrm{i}}\,, \qquad s(-k) = -s(k)\,, \qquad s(k) \in \mathbb{R}\,. \tag{2.30}$$

Under mild technical assumptions on $s$, one can define the pseudo-differential operator

$$\left[D_s f\right](x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} s(k) \hat{f}(k) \mathrm{e}^{-\mathrm{i}kx}\,, \tag{2.31}$$

$\hat{f}$ being the Fourier transform of $f$. The function $s$ is called the symbol of the operator $D_s$. Setting $\mathrm{e}_p(x) \equiv \mathrm{e}^{-\mathrm{i}px}$, one immediately verifies that

$$\left[D_s \mathrm{e}_p\right](x) = s(p)\mathrm{e}_p(x)\,, \qquad \left[D_s \mathrm{e}_{-p}\right](x) = s(-p)\mathrm{e}_{-p}(x) = -s(p)\mathrm{e}_{-p}(x)\,. \tag{2.32}$$

Using these simple identities and the representation (2.16) with $S$ given by (2.30), one obtains that $\psi \equiv \psi_1$ satisfies the general boundary condition [14]

$$\left[\left(D_s + \mathrm{i}\right)\psi\right](t, 0) = 0\,. \tag{2.33}$$

The above argument extends to star graphs with $n > 1$ edges as well. One concludes therefore that besides (2.5), the algebra $\mathcal{A}$ covers also more general boundary conditions, formulated in terms of pseudo-differential operators. Computing the correlation functions of $\psi$ and $\psi^*$ in the next section, we apply the general framework offered by $\mathcal{A}$ without referring necessarily to (2.5).

It is well-known that the non-relativistic field $\psi$ can be quantized with Fermi statistics as well, replacing the commutators (2.2,2.3) with anticommutators. Obviously, for this purpose one has to perform the same modification in (2.19-2.22).

## 2.2 Relativistic dispersion

The above setting has a straightforward extension to quantum fields with relativistic dispersion relation. Let us consider for example the Hermitian scalar field $\varphi(t,x,i)$ defined on the graph $G$ by

$$\left(\partial_t^2 - \partial_x^2 + m^2\right)\varphi(t,x,i) = 0\,, \qquad t\in\mathbb{R}\,, \quad x>0\,, \quad i=1,...,n\,, \tag{2.34}$$

and the equal-time canonical commutation relations

$$[\varphi(0,x_1,i_1)\,,\,\varphi(0,x_2,i_2)] = 0\,, \tag{2.35}$$

$$[(\partial_t\varphi)(0,x_1,i_1)\,,\,\varphi(0,x_2,i_2)] = -i\delta_{i_1}^{i_2}\,\delta(x_1-x_2)\,. \tag{2.36}$$

Since $\varphi$ is Hermitian, we impose on it the counterpart of the vertex boundary condition (2.5) for real $A$ and $B$. In this case one infers from (2.8) that $S$ is symmetric, i. e.

$$S^t(k) = S(k)\,. \tag{2.37}$$

Like in the non-relativistic case, we introduce at this stage the algebra $\mathcal{A}$ with $S$ obeying (2.9,2.10,2.14,2.37). Then

$$\varphi(t,x,i) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi\sqrt{2\omega(k)}}\left[a^{*i}(k)\mathrm{e}^{\mathrm{i}\omega(k)t-\mathrm{i}kx} + a_i(k)\mathrm{e}^{-\mathrm{i}\omega(k)t+\mathrm{i}kx}\right]\,, \tag{2.38}$$

with

$$\omega(k) = \sqrt{k^2+m^2}\,, \tag{2.39}$$

is the solution of (2.34-2.36) we are looking for. The Hamiltonian is obtained from (2.27), inserting the relativistic dispersion relation (2.39).

Summarizing, we introduced canonical quantum fields on a star graph $G$. The new feature, with respect to the conventional free fields on $\mathbb{R}^s$, is the presence of a non-trivial one-particle $S$-matrix describing the interaction at the vertex $V$. The latter represents physically a point-like defect. Quantum field theory with such defects has been investigated [15]-[20] in various frameworks. The results of this section confirm once more the conclusion of [21], indicating the algebra $\mathcal{A}$ as a universal tool for quantization with defects. Our goal in what follows will be to investigate the basic features of the quantum fields constructed so far on $G$. Before doing that however, it is instructive to illustrate at this stage the vertex boundary conditions (2.5) with some examples.

## 2.3 Examples

The simplest choice for $A$ and $B$ in (2.5) is

$$A = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad B = 0\,, \tag{2.40}$$

which give rise to the star graph generalization

$$\psi_1(t,0) = \psi_2(t,0) = \ldots = \psi_n(t,0) = 0 \tag{2.41}$$

of the familiar Dirichlet boundary condition. In this case $S(k) = \mathbb{I}_n$.

A slightly more complicated example is

$$A = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -\eta \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \tag{2.42}$$

which leads to

$$\psi_1(t,0) = \psi_2(t,0) = \ldots = \psi_n(t,0)\,, \qquad \sum_{i=1}^{n} (\partial_x \psi_i)(t,0) = \eta \psi_n(t,0)\,, \tag{2.43}$$

generalizing the Robin (mixed) boundary condition. The associated $S$-matrix is non-trivial,

$$S(k) = \frac{1}{nk + \mathrm{i}\eta} \begin{pmatrix} (2-n)k - \mathrm{i}\eta & 2k & 2k & \cdots & 2k \\ 2k & (2-n)k - \mathrm{i}\eta & 2k & \cdots & 2k \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2k & 2k & 2k & \cdots & (2-n)k - \mathrm{i}\eta \end{pmatrix}. \tag{2.44}$$

In particular, $\eta = 0$ provides a generalization of the Neumann condition for $n \geq 2$.

Notice that the boundary conditions (2.41,2.43) are symmetric under edge permutations. This is clearly not the case in general. A simple asymmetric example is defined by

$$A = \frac{2\lambda}{3}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda > 0\,, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \tag{2.45}$$

The associated $S$-matrix reads

$$S(k) = \frac{1}{3(k+\mathrm{i}\lambda)}\begin{pmatrix} 3k - \mathrm{i}\lambda & 2\mathrm{i}\lambda & 2\mathrm{i}\lambda \\ 2\mathrm{i}\lambda & -\mathrm{i}\lambda & 3k + 2\mathrm{i}\lambda \\ 2\mathrm{i}\lambda & 3k + 2\mathrm{i}\lambda & -\mathrm{i}\lambda \end{pmatrix}, \tag{2.46}$$

which shows that the boundary condition determined by (2.45) is not invariant under the permutations $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$.

Being associated with critical points in the context of statistical mechanics, the scale invariant vertex boundary conditions are of special interest. The subset of such conditions of the form (2.5) has been fully described in [3]. As expected, the corresponding $S$-matrices are $k$-independent. An example of this type, obtained by setting $\eta = 0$ in (2.44), is

$$S = \frac{1}{n}\begin{pmatrix} (2-n) & 2 & 2 & \cdots & 2 \\ 2 & (2-n) & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2 & 2 & 2 & \cdots & (2-n) \end{pmatrix}. \tag{2.47}$$

Let us observe finally that given a pair of matrices $(A, B)$ fixing the boundary condition (2.5), the pair $(\widetilde{A}, \widetilde{B}) = (-B, A)$ defines an admissible condition as well. By means of (2.8), the relative scattering matrix satisfies

$$\widetilde{S}(k) = S(k^{-1})\,. \tag{2.48}$$

This remarkable property [3] shows that the mapping $(A, B) \mapsto (-B, A)$ is a sort of duality transformation, relating high and low energies. Being $k$-independent, the scale invariant $S$-matrices are obviously dual invariant. One should keep in mind that duality does not preserve in general the completeness condition (2.14). In fact, it follows from (2.48) that resonant states are dual images of bound states and vice versa.

# 3 Correlation functions

In order to extract concrete physical information from the algebra $\mathcal{A}$, one should study its representations. We focus here on two of them: the Fock representation $\mathcal{F}(\mathcal{A})$ and the Gibbs representation $\mathcal{G}_\beta(\mathcal{A})$ at inverse temperature $\beta$. Since all technical details about the construction and the properties of $\mathcal{F}(\mathcal{A})$ and $\mathcal{G}_\beta(\mathcal{A})$ can be found in [10] and [13] respectively, we collect below only the basic formulae. Employing the commutation relations (2.19,2.20,2.22), one can reduce the computation of a generic correlation function to correlators of the form

$$\langle \prod_{k=1}^{m} a_{i_k}(p_{i_k}) \prod_{l=1}^{n} a^{*j_l}(q_{j_l})\rangle \, , \tag{3.1}$$

which can be evaluated in turn by iteration via

$$\langle \prod_{k=1}^{m} a_{i_k}(p_{i_k}) \prod_{l=1}^{n} a^{*j_l}(q_{j_l})\rangle = \delta_{mn} \sum_{k=1}^{m} \langle a_{i_1}(p_{i_1}) a^{*j_k}(q_{j_k})\rangle \, \langle \prod_{k=2}^{m} a_{i_k}(p_{i_k}) \prod_{\substack{l=1 \\ l\neq k}}^{n} a^{*j_k}(q_{j_k})\rangle \, . \tag{3.2}$$

For this purpose one needs only the two-point functions. In $\mathcal{F}(\mathcal{A})$ on has

$$\langle a_i(p) a^{*j}(q)\rangle = 2\pi \left[\delta_i^j\, \delta(p-q) + S_i^j(p)\, \delta(p+q)\right] \, , \qquad \langle a^{*i}(p) a_j(q)\rangle = 0\, . \tag{3.3}$$

In $\mathcal{G}_\beta(\mathcal{A})$ one finds [13] instead

$$\langle a_i(p) a^{*j}(q)\rangle_\beta = \frac{1}{1 \pm \mathrm{e}^{-\beta[\omega(p)-\mu]}}\, 2\pi \left[\delta_i^j \delta(p-q) + S_i^j(p)\delta(p+q)\right] \, , \tag{3.4}$$

$$\langle a^{*i}(p) a_j(q)\rangle_\beta = \frac{\mathrm{e}^{-\beta[\omega(p)-\mu]}}{1 \pm \mathrm{e}^{-\beta[\omega(p)-\mu]}}\, 2\pi \left[\delta_j^i \delta(p-q) + S_j^i(-p)\delta(p+q)\right] \, , \tag{3.5}$$

where $\pm$ stands for Fermi/Bose statistics and $\mu \in \mathbb{R}$ is the chemical potential associated with the number operator

$$N = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dk}{2\pi} a^{*i}(k) a_i(k)\, . \tag{3.6}$$

A characteristic property of any thermal representation is the Kubo-Martin-Schwinger (KMS) condition. In our context it states that

$$\langle [\alpha_s a_i(p)]\, a^{*j}(q)\rangle_\beta = \langle a^{*j}(q)\, [\alpha_{s+\mathrm{i}\beta} a_i(p)]\rangle_\beta \, , \tag{3.7}$$

$\alpha_s$ being the automorphism

$$\alpha_s\, a^{*i}(k) = \mathrm{e}^{\mathrm{i}sK} a^{*i}(k) \mathrm{e}^{-\mathrm{i}sK}\, , \qquad \alpha_s\, a_i(k) = \mathrm{e}^{\mathrm{i}sK} a_i(k) \mathrm{e}^{-\mathrm{i}sK}\, , \qquad K \equiv H - \mu N\, . \tag{3.8}$$

One easily verifies that (3.4,3.5) satisfy (3.7).

It is straightforward now to derive the two-point correlators of the fields $\psi$, $\psi^*$ and $\varphi$ introduced in the previous section. In the Fock representation $\mathcal{F}(\mathcal{A})$ one gets

$$\langle \psi^{*i_1}(t_1,x_1)\psi_{i_2}(t_2,x_2)\rangle = \\ \langle \psi_{i_1}(t_1,x_1)\psi_{i_2}(t_2,x_2)\rangle = \langle \psi^{*i_1}(t_1,x_1)\psi^{*i_2}(t_2,x_2)\rangle = 0\,, \tag{3.9}$$

$$\langle \psi_{i_1}(t_1,x_1)\psi^{*i_2}(t_2,x_2)\rangle = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\mathrm{e}^{-\mathrm{i}\omega(k)t_{12}}\left[\mathrm{e}^{\mathrm{i}kx_{12}}\delta_{i_1}^{i_2}+\mathrm{e}^{\mathrm{i}k\widetilde{x}_{12}}S_{i_1}^{i_2}(k)\right]\,, \tag{3.10}$$

with

$$x_{12}\equiv x_1-x_2\,,\quad \widetilde{x}_{12}\equiv x_1+x_2 \tag{3.11}$$

and the dispersion relation (2.18). In the relativistic case (2.39) one obtains

$$\langle \varphi(t_1,x_1,i_1)\varphi(t_2,x_2,i_2)\rangle = \int_{-\infty}^{+\infty}\frac{dk}{4\pi\omega(k)}\mathrm{e}^{-\mathrm{i}\omega(k)t_{12}}\left[\mathrm{e}^{\mathrm{i}kx_{12}}\delta_{i_1}^{i_2}+\mathrm{e}^{\mathrm{i}k\widetilde{x}_{12}}S_{i_1}^{i_2}(k)\right]\,. \tag{3.12}$$

We see that the fields in the different edges are not independent, but interact through the scattering matrix $S$ taking into account the boundary conditions in the vertex of the graph.

In the Gibbs representation $\mathcal{G}_\beta(\mathcal{A})$ one finds[5]

$$\langle \psi_{i_1}(t_1,x_1)\psi_{i_2}(t_2,x_2)\rangle_\beta = \langle \psi^{*i_1}(t_1,x_1)\psi^{*i_2}(t_2,x_2)\rangle_\beta = 0\,, \tag{3.13}$$

$$\langle \psi_{i_1}(t_1,x_1)\psi^{*i_2}(t_2,x_2)\rangle_\beta = \\ \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{1}{1\pm \mathrm{e}^{-\beta[\omega(k)-\mu]}}\mathrm{e}^{-\mathrm{i}\omega(k)t_{12}}\left[\mathrm{e}^{\mathrm{i}kx_{12}}\delta_{i_1}^{i_2}+\mathrm{e}^{\mathrm{i}k\widetilde{x}_{12}}S_{i_1}^{i_2}(k)\right]\,, \tag{3.14}$$

$$\langle \psi^{*i_1}(t_1,x_1)\psi_{i_2}(t_2,x_2)\rangle_\beta = \\ \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{\mathrm{e}^{-\beta[\omega(k)-\mu]}}{1\pm \mathrm{e}^{-\beta[\omega(k)-\mu]}}\mathrm{e}^{\mathrm{i}\omega(k)t_{12}}\left[\mathrm{e}^{\mathrm{i}kx_{12}}\delta_{i_2}^{i_1}+\mathrm{e}^{\mathrm{i}k\widetilde{x}_{12}}S_{i_2}^{i_1}(k)\right]\,, \tag{3.15}$$

$$\langle \varphi(t_1,x_1,i_1)\varphi(t_2,x_2,i_2)\rangle_\beta = \\ \int_{-\infty}^{+\infty}\frac{dk}{4\pi\omega(k)}\frac{\mathrm{e}^{-\beta[\omega(k)-\mu]+i\omega(k)t_{12}}+\mathrm{e}^{-i\omega(k)t_{12}}}{1-\mathrm{e}^{-\beta[\omega(k)-\mu]}}\left[\mathrm{e}^{\mathrm{i}kx_{12}}\delta_{i_1}^{i_2}+\mathrm{e}^{\mathrm{i}k\widetilde{x}_{12}}S_{i_1}^{i_2}(k)\right]\,. \tag{3.16}$$

With the above background we are ready to compute some quantum field theory observables like the particle, current and energy densities on a star graph.

[5] In the bosonic case we assume that $\mu < 0$.

## 4 Observables

The basic local observables associated with the field $\psi$ are the particle density

$$\varrho(t,x,i) = \left[\psi^{*i}\psi_i\right](t,x)\,, \tag{4.1}$$

the current density

$$j(t,x,i) = \frac{\mathrm{i}}{2m}\left[\psi^{*i}(\partial_x\psi_i) - (\partial_x\psi^{*i})\psi_i\right](t,x)\,, \tag{4.2}$$

and the energy density

$$\theta(t,x,i) = -\frac{1}{4m}\left[\psi^{*i}\left(\partial_x^2\psi_i\right) + \left(\partial_x^2\psi^{*i}\right)\psi_i\right](t,x)\,, \tag{4.3}$$

without summation over $i$ in the right hand side. Conservation and Kirchoff's laws

$$\partial_t\,\varrho(t,x,i) = \partial_x\, j(t,x,i)\,, \qquad \sum_{i=1}^n j(t,0,i) = 0\,, \tag{4.4}$$

follow from (2.1) and (2.5) respectively.

It is straightforward to compute the expectation values of (4.1-4.3) in the Gibbs representation $\mathcal{G}_\beta(\mathcal{A})$ defined in the previous section. Applying the conventional point-splitting procedure, one gets from (3.15)

$$\langle\varrho(t,x,i)\rangle_\beta = \lim_{\substack{t_1\to t_2=t\\ x_1\to x_2=x}} \langle\psi^{*i_1}(t_1,x_1)\psi_{i_2}(t_2,x_2)\rangle_\beta =$$

$$\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\frac{e^{-\beta[\omega(k)-\mu]}}{1\pm e^{-\beta[\omega(k)-\mu]}}[1+S_i^i(k)e^{2\mathrm{i}kx}]\,, \qquad \omega(k) = \frac{k^2}{2m}\,. \tag{4.5}$$

Analogously, $\langle j(t,x,i)\rangle_\beta = 0$ and

$$\langle\theta(t,x,i)\rangle_\beta = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\omega(k)\,\frac{e^{-\beta[\omega(k)-\mu]}}{1\pm e^{-\beta[\omega(k)-\mu]}}[1+S_i^i(k)e^{2\mathrm{i}kx}]\,. \tag{4.6}$$

Egs. (4.5,4.6) provide convenient integral representations of the particle and energy densities in the Gibbs state. Since the integrands involve only diagonal elements of the $S$-matrix, the expectation values (4.5,4.6) are real because of Hermitian analyticity (2.10). In agreement with the invariance of the Gibbs state under time translations, (4.5,4.6) are $t$-independent and vanish in the limit of zero temperature ($\beta\to\infty$), which shows that these densities have a purely thermal origin.

Let us concentrate on the energy density (4.6). It is instructive to separate the universal contribution

$$\varepsilon_\pm(\beta) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, \omega(k)\, \frac{e^{-\beta[\omega(k)-\mu]}}{1 \pm e^{-\beta[\omega(k)-\mu]}}\, , \tag{4.7}$$

from the vertex dependent part

$$\mathcal{E}_\pm(x,i,\beta) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, \omega(k)\, \frac{e^{-\beta[\omega(k)-\mu]}}{1 \pm e^{-\beta[\omega(k)-\mu]}}\, S_i^i(k) e^{2ikx}\, . \tag{4.8}$$

Unitarity (2.9) of $S$ implies the estimate

$$|\mathcal{E}_\pm(x,i,\beta)| \leq \varepsilon_\pm(\beta)\, , \tag{4.9}$$

leading in turn to positivity

$$\langle \theta(t,x,i)\rangle_\beta = \varepsilon_\pm(\beta) + \mathcal{E}_\pm(x,i,\beta) \geq 0 \tag{4.10}$$

of the energy density in the Gibbs state. The integration over $k$ in (4.7) gives

$$\varepsilon_\pm(\beta) = \mp \frac{1}{2m\sqrt{2\pi}} \mathrm{Li}_{\frac{3}{2}}\left(\mp \mathrm{e}^{\beta\mu}\right) \left(\frac{m}{\beta}\right)^{\frac{3}{2}}\, , \tag{4.11}$$

where $\mathrm{Li}_s$ is the polylogarithm function. Eq. (4.11) is the analog of the Stefan-Boltzmann (S-B) law for the non-relativistic dispersion relation (2.18). With $\mu = 0$ it simplifies further

$$\varepsilon_-(\beta) = \frac{1}{2m\sqrt{2\pi}}\, \zeta\left(\frac{3}{2}\right) \left(\frac{m}{\beta}\right)^{\frac{3}{2}}\, , \quad \varepsilon_+(\beta) = \frac{\sqrt{2}-1}{2m\sqrt{2\pi}}\, \zeta\left(\frac{3}{2}\right) \left(\frac{m}{\beta}\right)^{\frac{3}{2}}\, , \tag{4.12}$$

$\zeta$ being Riemann's zeta function.

According to the previous discussion, the vertex dependent contribution (4.8) represents a correction to the S-B law generated by the defect at $x = 0$. As expected, this term is both $x$ and $\beta$-dependent and involves the $S$-matrix. The explicit integration in $k$ is usually a hard problem, but the integral representation (4.8) is very convenient for the numerical evaluation of $\mathcal{E}_\pm(x,i,\beta)$.

Let us consider now the case with relativistic dispersion (2.39), focusing on the scalar field (2.34-2.36). The energy density in this case reads

$$\theta(t,x,i) = \frac{1}{2}\left\{(\partial_t\varphi)^2 - \frac{1}{2}\left[\varphi(\partial_x^2\varphi) + (\partial_x^2\varphi)\varphi\right] + m^2\varphi^2\right\}(t,x,i)\, . \tag{4.13}$$

One can compute the expectation value $\langle\theta(t,x,i)\rangle_\beta$ via point-splitting from the two-point function (3.16). Differently from the non-relativistic case, it turns out that $\langle\theta(t,x,i)\rangle_\beta$ diverges. This is not surprising because the same problem is encountered already in the standard Casimir effect. Like in that case, one can solve it by subtracting from $\langle\theta(t,x,i)\rangle_\beta$ the expectation value $\langle\theta(t,x)\rangle_\infty^{\rm line}$ of the energy density $\theta(t,x)$ on the whole line $\mathbb{R}$ and at zero temperature. The result is

$$\langle\theta(t,x,i)\rangle_\beta - \langle\theta(t,x)\rangle_\infty^{\rm line} = \varepsilon_{_{\rm S-B}}(\beta) + \mathcal{E}_{_{\rm C}}(x,i) + \mathcal{E}(x,i,\beta)\, , \tag{4.14}$$

where

$$\varepsilon_{_{\rm S-B}}(\beta) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, \omega(k)\, \frac{e^{-\beta[\omega(k)-\mu]}}{1-e^{-\beta[\omega(k)-\mu]}}\, , \qquad \omega(k) = \sqrt{k^2+m^2}\, , \tag{4.15}$$

is the Stefan-Boltzmann contribution,

$$\mathcal{E}_{_{\rm C}}(x,i) = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, \omega(k)\, S_i^i(k) e^{2{\rm i}kx} \tag{4.16}$$

is the Casimir energy density associated with the vertex interaction at zero temperature and

$$\mathcal{E}(x,i,\beta) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, \omega(k)\, \frac{e^{-\beta[\omega(k)-\mu]}}{1-e^{-\beta[\omega(k)-\mu]}}\, S_i^i(k) e^{2{\rm i}kx} \tag{4.17}$$

is the finite temperature correction to the latter.

It is now easy to analyze separately the three contributions (4.15-4.17). To the end of this section we set for simplicity $m=\mu=0$. Then

$$\varepsilon_{_{\rm S-B}}(\beta) = \frac{\pi}{6\beta^2}\, , \tag{4.18}$$

which is the familiar S-B thermal energy density for a massless Hermitian scalar field in 1+1 space-time dimensions.

Eq. (4.16) is a useful integral representation of the Casimir energy density in terms of the scattering matrix $S(k)$. This representation extends the result of [9] to all boundary conditions of the type (2.5). In the scale invariant case for instance, $S(k)$ is a constant matrix $S$ leading to

$$\mathcal{E}_{_{\rm C}}(x,i) = -\frac{S_i^i}{8\pi x^2}\, . \tag{4.19}$$

The corresponding finite temperature correction (4.17) reads

$$\mathcal{E}(x,i,\beta) = \frac{\pi S_i^i}{2\beta^2 \sinh\left(2\pi\frac{x}{\beta}\right)} - \frac{S_i^i}{8\pi x^2}\, . \tag{4.20}$$

We illustrate the case of a $k$-dependent $S$-matrix on the example (2.45) of asymmetric boundary conditions in the vertex $V$. The $k$-integration in (4.16) with the diagonal elements of (2.46) can be performed explicitly and one finds

$$\mathcal{E}_{\mathrm{C}}(x,1) = -\frac{1}{8\pi x^2} + \frac{\lambda}{3\pi x} + \frac{2\lambda^2 \mathrm{e}^{2\lambda x}}{3\pi}\,\mathrm{Ei}(-2\lambda x)\,, \tag{4.21}$$

$$\mathcal{E}_{\mathrm{C}}(x,i) = \frac{\lambda}{12\pi x} + \frac{\lambda^2 \mathrm{e}^{2\lambda x}}{6\pi}\,\mathrm{Ei}(-2\lambda x)\,, \qquad i=2,3\,, \tag{4.22}$$

Ei being the exponential-integral function. We have plotted these functions for $\lambda = 1$ in Fig. 2, which shows that the Casimir energy in the edge $E_1$ behaves quite differently from that in the edges $E_{2,3}$. In fact, close to the vertex along $E_1$ one has an attractive force, whereas the force along $E_{2,3}$ is always repulsive.

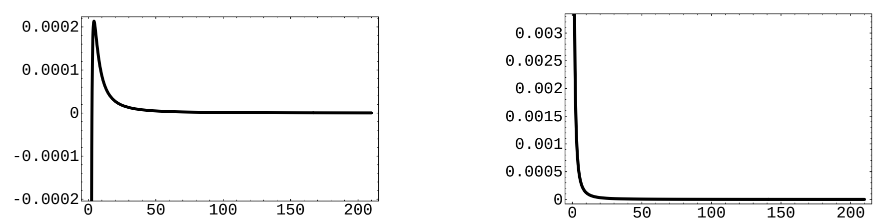


Figure 2: Plots of $\mathcal{E}_{\mathrm{C}}(x,1)$ (left) and $\mathcal{E}_{\mathrm{C}}(x,2) = \mathcal{E}_{\mathrm{C}}(x,3)$ (right) at $\lambda = 1$.

Using the results of [13], the thermal correction (4.17) in this case equals

$$\mathcal{E}(x,1,\beta) = \frac{\pi}{2\beta^2 \sinh\left(2\pi\frac{x}{\beta}\right)} - \frac{1}{8\pi x^2} - \frac{\lambda}{3\pi x} - \frac{2\lambda^2 \mathrm{e}^{2x\lambda}}{3\pi}\,\mathrm{Ei}(-2\lambda x) + \frac{4}{\beta^2} F\left(\frac{\beta\lambda}{2\pi}, \mathrm{e}^{-4\pi\frac{x}{\beta}}\right)\,, \tag{4.23}$$

$$\mathcal{E}(x,i,\beta) = -\frac{\lambda}{12\pi x} - \frac{\lambda^2 \mathrm{e}^{2x\lambda}}{6\pi}\,\mathrm{Ei}(-2\lambda x) + \frac{1}{\beta^2} F\left(\frac{\beta\lambda}{2\pi}, \mathrm{e}^{-4\pi\frac{x}{\beta}}\right)\,, \qquad i=2,3\,, \tag{4.24}$$

with

$$F(\sigma,\tau) = \frac{2\pi\sigma\tau}{3(1+\sigma)}\ {}_2\mathrm{F}_1\left[2, 1+\sigma; 2+\sigma; \tau\right]\,, \tag{4.25}$$

where ${}_2\mathrm{F}_1$ is the hypergeometric function. Fig. 3 and Fig. 4 display the behavior of $\mathcal{E}(x,i,\beta)$ at fixed $\beta$ and $x$ respectively.

Summarizing, we derived in this section the expectation values of some local observables in the Gibbs state, paying attention to the contribution of the vertex interaction captured by the boundary conditions (2.5). In what follows we will address the problem of bulk interactions.

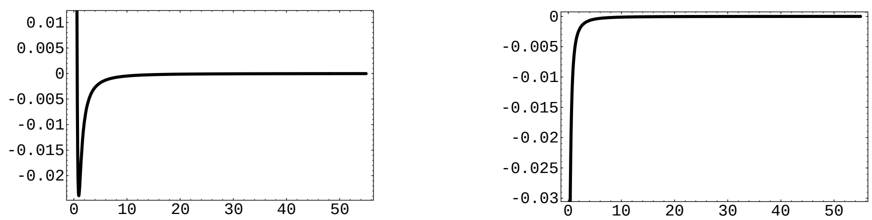


Figure 3: Plots of $\mathcal{E}(x,1,1)$ (left) and $\mathcal{E}(x,2,1)=\mathcal{E}(x,3,1)$ (right) at $\lambda=1$.

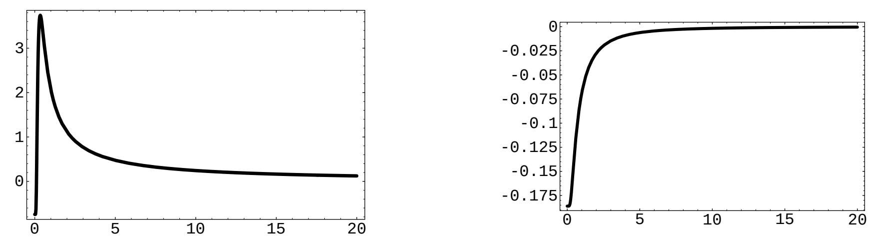


Figure 4: Plots of $\mathcal{E}(0.1,1,\beta)$ (left) and $\mathcal{E}(0.1,2,\beta)=\mathcal{E}(0.1,3,\beta)$ (right) at $\lambda=1$.

# 5 Non-trivial bulk interactions

The dynamics of the classical Non-Linear Schrödinger (NLS) model on a star graph $G$ is described by the equation of motion[6]

$$\left(\mathrm{i}\partial_t + \partial_x^2\right)\psi_i(t,x) = 2g\,|\psi_i(t,x)|^2\,\psi_i(t,x)\,, \tag{5.1}$$

$g\in\mathbb{R}$ being the coupling constant. In order to avoid the presence of bound states in the bulk, we take below $g>0$. Our main goal now will be to quantize (5.1) at zero temperature, imposing the initial conditions (2.2,2.3) and the boundary condition (2.5). For this purpose one must:

(i) construct a Hilbert space $\mathcal{H}$ describing the states of the system;

(ii) define on an appropriate dense domain $\mathcal{D}\subset\mathcal{H}$ the operator valued distributions $\psi_i(t,x)$ and $\psi^{*i}(t,x)$, satisfying (2.2,2.3,2.5) and the equation of motion (5.1) with suitably defined operator product in the right hand side;

(iii) exhibit a vacuum state $\Omega\in\mathcal{D}$, which is cyclic with respect to the fields $\psi^{*i}(t,x)$.

This program has been carried out for $n=1$ and $n=2$ in [22] and [23] respectively. Since the approach used there has a direct generalization to a star graph $G$

[6] In order to simplify the notation, in this section we fix the units in such a way that $m=\frac{1}{2}$.

with arbitrary $n$, we will sketch only the main points of the construction. The first step is to generalize the algebra $\mathcal{A}$ in order to incorporate both non-trivial bulk and vertex interactions. Following [24]-[28], we introduce the two-body bulk scattering matrix[7]

$$R_{12}(k) = \frac{k - \mathrm{i}g}{k + \mathrm{i}g}\, \mathbb{I}_n \otimes \mathbb{I}_n \,, \tag{5.2}$$

which satisfies unitarity

$$R_{12}(k)\, R_{21}(-k) = \mathbb{I}_n \otimes \mathbb{I}_n \,, \tag{5.3}$$

Hermitian analyticity

$$R^*_{12}(k) = R_{21}(-k)\,, \tag{5.4}$$

and the quantum Yang-Baxter equation

$$R_{12}(k_1-k_2)R_{13}(k_1-k_3)R_{23}(k_2-k_3) = R_{23}(k_2-k_3)R_{13}(k_1-k_3)R_{12}(k_1-k_2)\,. \tag{5.5}$$

In the spirit of factorized scattering [24]-[28] we replace (2.19,2.20) by

$$a_{i_1}(k_1)\; a_{i_2}(k_2) \;-\; R_{i_2 i_1}^{j_2 j_1}(k_2-k_1)\; a_{j_2}(k_2)\, a_{j_1}(k_1) = 0\,, \tag{5.6}$$

$$a^{*i_1}(k_1)\, a^{*i_2}(k_2) - a^{*j_2}(k_2)\, a^{*j_1}(k_1)\, R^{i_2 i_1}_{j_2 j_1}(k_2-k_1) = 0\,. \tag{5.7}$$

encoding in this way the bulk interaction in the exchange relations. On the other hand, in order to describe the vertex interaction we introduce [10] the so called *boundary* generators $b_i^j(k)$, which appear in the right hand side of

$$\begin{aligned} a_{i_1}(k_1)\, a^{*i_2}(k_2) \;-\; a^{*j_2}(k_2)\, R_{i_1 j_2}^{j_1 i_2}(k_1-k_2)\, a_{j_1}(k_1) \;&= \\ 2\pi\, \delta(k_1-k_2)\, \delta_{i_1}^{i_2}\, \mathbf{1} + 2\pi\, \delta(k_1+k_2)\, b_{i_1}^{i_2}(k_1)\,, \end{aligned} \tag{5.8}$$

being the counterpart of (2.21). Due to the particular form (5.2) of $R_{12}$, the boundary generators commute

$$[b_{i_1}^{j_1}(k_1)\,,\, b_{i_2}^{j_2}(k_2)] = 0\,, \tag{5.9}$$

but have non-trivial exchange relations with $a_i(k)$ and $a^{*i}(k)$:

$$a_{i_1}(k_1)\, b_{i_2}^{j_2}(k_2) = R_{i_2 i_1}^{k_2 k_1}(k_2-k_1)\, b_{k_2}^{l_2}(k_2)\, R_{k_1 l_2}^{l_1 j_2}(k_1+k_2)\, a_{l_1}(k_1)\,, \tag{5.10}$$

$$b_{i_1}^{j_1}(k_1)\, a^{*i_2}(k_2) = a^{*l_2}(k_2)\, R_{i_1 l_2}^{l_1 k_2}(k_1-k_2)\, b_{l_1}^{k_1}(k_1)\, R_{k_2 k_1}^{i_2 j_1}(k_2+k_1)\;. \tag{5.11}$$

The next step is to focus on the Fock representation $\mathcal{F}(\mathcal{A})$ of the above algebra $\mathcal{A}$. According to [10], $\mathcal{F}(\mathcal{A})$ is fixed (up to unitary equivalence) by the condensate

[7] We adopt the conventional tensor notation.

$\langle\Omega\,,\, b_i^j(k)\Omega\rangle$, where $\Omega$ is the vacuum state and $\langle\cdot\,,\,\cdot\rangle$ the scalar product in $\mathcal{F}(\mathcal{A})$. In order to satisfy the boundary condition (2.5), we require that

$$\langle\Omega\,,\, b_i^j(k)\Omega\rangle = S_i^j(k) \tag{5.12}$$

with $S(k)$ given by (2.7).

The final step is to define the basic structures, described in (i)-(iii) above, in terms of $\mathcal{F}(\mathcal{A})$ as follows:

- $\mathcal{H}$ and $\Omega$ are the Hilbert space and the vacuum state of $\mathcal{F}(\mathcal{A})$.
- The quantum field $\psi_i(t,x)$ admits the series representation

  $$\psi_i(t,x) = \sum_{m=0}^{\infty}(-g)^m\, \psi_i^{(m)}(t,x)\,, \tag{5.13}$$

  where

  $$\psi_i^{(m)}(t,x) = \int_{-\infty}^{\infty}\prod_{\substack{k=1\\ l=0}}^{m}\frac{dp_k}{2\pi}\frac{dq_l}{2\pi}\, a^{*i}(p_1)\ldots a^{*i}(p_m)a_i(q_m)\ldots a_i(q_0)\cdot$$
  $$\frac{e^{\mathrm{i}\sum\limits_{l=0}^{m}(q_l x - q_l^2 t) - \mathrm{i}\sum\limits_{k=1}^{m}(p_k x - p_k^2 t)}}{\prod\limits_{k=1}^{m}(p_k - q_{k-1} - \mathrm{i}\varepsilon)(p_k - q_k - \mathrm{i}\varepsilon)}\,, \tag{5.14}$$

  *without* summation over the upper and lower index $i$ in the right hand side.
- The domain $\mathcal{D}$ is the finite particle subspace of $\mathcal{F}(\mathcal{A})$, which is well-known to be dense in $\mathcal{H}$.

It is worth mentioning that the series (5.13) is actually a finite sum when $\psi_i(t,x)$ is acting on $\mathcal{D}$. The coupling constant $g$ appears explicitly in (5.13) and implicitly in $a_i(k)$ and $a^{*i}(k)$, which depend on $g$ through the bulk scattering matrix (5.2). Eq. (5.14) defines a sort of non-linear Fourier transform of the type used in [29, 30] to solve the classical NLS model. In oder to give meaning of the equation of motion (5.1) at the level of quantum fields we perform the substitution

$$|\psi_i(t,x)|^2\psi_i(t,x)\ \mapsto\ :\psi_i\psi^{*i}\psi_i:(t,x)\,, \tag{5.15}$$

(without summation over $i$) introducing the normal ordered product $:\cdots:$ as follows [22]: all creation operators in such a product stand as usual to the left of all annihilation operators. In view of eqs. (5.6,5.7), in our case one must further specify the ordering of creators and annihilators themselves. We define $: \cdots :$ to preserve the original order of the creators. The original order of the annihilators is preserved if both belong to the same $\psi$ or $\psi^*$ and inverted otherwise. With this convention one can show [22] that

$$(i\partial_t + \partial_x^2)\langle \chi_1\, ,\, \psi_i(t,x)\chi_2\rangle = 2g\, \langle \chi_1\, ,\, :\psi_i \psi^{*i} \psi_i : (t,x)\chi_2\rangle\, , \quad x > 0,\ i = 1, ..., n, \tag{5.16}$$

holds for any $\chi_{1,2} \in \mathcal{D}$. The boundary condition (2.5) is also satisfied in mean value on $\mathcal{D}$, namely

$$\langle \chi_1\, ,\, \left[A_i^j \psi_j(t,0) + B_i^j (\partial_x \psi_j)(t,0)\right] \chi_2\rangle = 0\, , \quad \forall\, t \in \mathbb{R}\, , \quad \chi_{1,2} \in \mathcal{D}\, . \tag{5.17}$$

Finally, by means of (5.6-5.11) one can verify the commutation relations (2.2,2.3). Since $\psi$ and $\psi^*$ are unbounded operators, the subtle points in proving the above statements are essentially domain problems. They are faced [22] using the absence of poles in the analytic extension of the boundary and the bulk scattering matrices (2.7) and (5.2) in the upper complex $k$-plane. The lack of both bulk ($g > 0$) and vertex (see eq.(2.14)) bound states is therefore crucial.

The representation defined by (5.13,5.14) allows to compute all correlation functions. From the general structure of the solution one infers that the nonvanishing correlators involve equal number of $\psi_i$ and $\psi^{*i}$. Moreover, for deriving the exact $2n$-point function one does not need all terms in the expansion (5.13), but at most the $(n-1)$-th order contribution. In fact,

$$\langle \psi_{i_1}(t_1,x_1)\psi^{*i_2}(t_2,x_2)\rangle = \langle \psi_{i_1}^{(0)}(t_1,x_1)\psi^{*i_2(0)}(t_2,x_2)\rangle\, , \tag{5.18}$$

$$\begin{gathered} \langle \psi_{i_1}(t_1,x_1)\psi_{i_2}(t_2,x_2)\psi^{*i_3}(t_3,x_3)\psi^{*i_4}(t_4,x_4)\rangle = \\ \langle \psi_{i_1}^{(0)}(t_1,x_1)\psi_{i_2}^{(0)}(t_2,x_2)\psi^{*i_3(0)}(t_3,x_3)\psi^{*i_4(0)}(t_4,x_4)\rangle + \\ g^2 \langle \psi_{i_1}^{(0)}(t_1,x_1)\psi_{i_2}^{(1)}(t_2,x_2)\psi^{*i_3(1)}(t_3,x_3)\psi^{*i_4(0)}(t_4,x_4)\rangle\, , \end{gathered} \tag{5.19}$$

and so on. Since the vacuum expectation value of any number of $\{a_i(k),\, a^{*i}(k),\, b_i^j(k)\}$ is known explicitly, one derives in this way an integral representation for the $2n$-point function involving $n$ momentum integrations. The correlation functions involving the particle density (4.1) and the current (4.2) can be computed analogously. It will be interesting to extend this framework to finite temperature.

The application of the above solution to the study of non-linear effects in the wave propagation on star graph quantum wires deserves further investigations.

# 6 Outlook and conclusions

We described in the present paper a construction of quantum fields propagating on a star graph $G$ with any number of edges. Our construction is based on a specific deformation $\mathcal{A}$ of the algebra of canonical commutation relations, which takes into account both vertex and bulk interactions. We consider the case without dissipation, leading to unitary theories. Two applications of the general framework have been discussed. The first one concerns the derivation of the expectation values of some observables at finite temperature. Focusing mainly on the energy density, we described the Casimir effect on a star graph $G$ and derived the correction to the Stefan-Boltzmann law due to the interaction in the vertex $V \in G$. As a second application we solved the quantum non-linear Schrödinger equation on $G$. This example illustrates our approach at work when integrable interactions are present in the bulk.

The results of the paper can be generalized in several directions. First of all one can increase the dimensions, setting $E_i = \mathbb{R}^s$ and $V = \mathbb{R}^{s-1}$ with $s > 1$. The case $s = 2$ gives rise to the so called quantum walls. In general, for $s > 1$ the fields propagating in $E_i$ induce a non-trivial quantum field theory on $V$, which is no longer a point. The critical behavior of such kind of theories, with interactions localized exclusively on $V$, has been investigated in [31]. On the other hand, phenomenological models with one edge and $s = 4$, aiming to construct effective theories of the fundamental particle interactions, have been introduced in [32] and still attract much attention. In this case the vertex $V = \mathbb{R}^3$ is interpreted as a 3-brane confining our universe.

Concerning the applications to condensed matter physics, it is essential to construct quantum field theory models describing the conductance properties of quantum wires. It will be interesting in this respect to extend the bosonization procedure and the vertex algebra construction of [33, 34] to star graphs with any number of edges. The case of general graphs with $N > 1$ vertices, which are thus modeling more realistic quantum wires, represents a challenging problem as well. We are currently investigating some of these issues.

# Appendix

The orthogonality relation (2.13) of the wave functions (2.12) is a consequence of the self-adjointness of the operator $-\partial_x^2$ on the graph $G$ with the boundary condition (2.5). In order check this property directly, it is convenient to introduce the $T$-matrix defined by

$$S(k) = \mathbb{I}_n + \mathrm{i}\, T(k)\,. \tag{6.1}$$

The explicit form of $T$, following from (2.6, 2.7, 2.8), is

$$T(k) = (A + \mathrm{i}kB)^{-1} 2\mathrm{i}A = 2\mathrm{i}A^*(A^* + \mathrm{i}kB^*)^{-1} \,. \tag{6.2}$$

The integral over $x \in \mathbb{R}_+$ in (2.13) can be computed by means of the well-known identity

$$-\,\mathrm{i}\int_0^{+\infty} dx\, \mathrm{e}^{\mathrm{i}kx} = \frac{1}{k + \mathrm{i}\varepsilon} = \frac{1}{k} - \mathrm{i}\pi\delta(k)\,, \tag{6.3}$$

the distribution $1/k$ being defined by means of the principal value prescription. One finds

$$\int_0^{+\infty} dx\, \chi_i^{*l}(x;k)\chi_l^j(x;p) = \delta_i^j\, 2\pi\, \delta(k-p) + \mathcal{T}_i^j(k,p)\,, \tag{6.4}$$

where

$$\mathcal{T}_i^j(k,p) = \frac{1}{p-k}[T_i^j(k) + T_i^j(-p) + \mathrm{i}\, T_i^l(k)T_l^j(-p)] - \frac{1}{p+k}[T_i^j(k) - T_i^j(-p)]\,. \tag{6.5}$$

The final step is to prove that the matrix $\mathcal{T}(k,p)$ vanishes. Using (6.2) one has in fact

$$\begin{aligned} T(k) + T(-p) \;&=\; 4\mathrm{i}A^*[AA^* + kpBB^* + \mathrm{i}(k-p)AB^*]^{-1}A + \\ &\phantom{=}\; 2pA^*[AA^* + kpBB^* + \mathrm{i}(k-p)AB^*]^{-1}B - \\ &\phantom{=}\; 2kB^*[AA^* + kpBB^* + \mathrm{i}(k-p)AB^*]^{-1}A\,, \end{aligned} \tag{6.6}$$

$$\begin{aligned} T(k) - T(-p) \;&=\; 2pA^*[AA^* + kpBB^* + \mathrm{i}(k-p)AB^*]^{-1}B + \\ &\phantom{=}\; 2kB^*[AA^* + kpBB^* + \mathrm{i}(k-p)AB^*]^{-1}A\,, \end{aligned} \tag{6.7}$$

$$T(k)T(-p) = -4A^*[AA^* + kpBB^* + \mathrm{i}(k-p)AB^*]^{-1}A\,, \tag{6.8}$$

which allow to rewrite $\mathcal{T}(k,p)$ in the form

$$\begin{aligned} \mathcal{T}(k,p) = \frac{4k}{p^2 - k^2}\{&pA^*[AA^* + kpBB^* + \mathrm{i}(k-p)AB^*]^{-1}B - \\ &pB^*[AA^* + kpBB^* + \mathrm{i}(k-p)AB^*]^{-1}A\}\,. \end{aligned} \tag{6.9}$$

The vanishing of (6.9) now follows from the condition (2.6), which implies

$$\begin{aligned} [AA^* + kpBB^* + \mathrm{i}(k-p)AB^*]^{-1} &= (A^* + \mathrm{i}kB^*)^{-1}(A - \mathrm{i}pB)^{-1} = \\ &\phantom{=}\;(A^* - \mathrm{i}pB^*)^{-1}(A + \mathrm{i}kB)^{-1}\,. \end{aligned} \tag{6.10}$$

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